Permutation Symmetries of Two-Body Amplitudes

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Considerations of permutation symmetry among the three channels in two-body scattering suggest that only amplitudes of symmetry S are nonvanishing in the forward direction at infinite energies; the other three-object symmetries A and D are not represented. For unlike scattering particles, this result is equivalent to an earlier one based on Pomeranchuk's theorem. For like particles the restrictions are much stronger and contradict the dominance of universal particle exchange at high energies; limiting cross sections will not generally be independent of isotopic spin, although this is accidentally true for NN scattering. Applications are here restricted mainly to isotopic spin, although an interesting point emerges: Particles with half-integer isotopic spin must also possess some other intrinsic parameter of spinor character.

1. INTRODUCTION AND SUMMARY

FOR any two-body process only two of the Mandelstam variables s, t, u are independent. It is often convenient to retain the redundant variable, however, and even to alter its choice in different regions. Any amplitude $a(stu)$ therefore has permutation symmetry appropriate to three variables: namely, S (totally symmetric), A (totally antisymmetric), and D (two-dimensional). A physical amplitude is generally characterized by some additional indices specific to the channel of the reaction, so we write a_s^α, a_t^β, a_u^γ. Exchange operations can then be represented by matrices $\mathcal{P}$ acting on the index α:

$$a_s^\alpha=\sum_\beta \mathcal{P}_{st}^{\alpha\beta}a_t^\beta. \qquad (1)$$

Here $a_s^\alpha(stu)$ and $a_t^\beta(stu)$ are generally different functions evaluated for identical arguments. The most immediate choice of α is to represent iostopic spin; we maintain this choice throughout, except in a couple of elementary applications.

If the matrices $\mathcal{P}$ are diagonalized, the corresponding a can be specified by symmetry: a^S, a^A, $a_s^\pm$. The completely symmetric representation is identical in all channels and the completely antisymmetric representation varies only by a sign, which is ignorable for our purposes; but for the two-dimensional representation the channel must be indicated. Restrictions on $a(stu)$ due to exchange symmetry are comparable in rigor to those imposed by analyticity and unitarity but may be independent in content. The present note examines this content, using the phase representation for forward scattering at high energies, and concludes that

the antisymmetric amplitude $a^A\to 0$ *as* $x\to\infty$, $y=0^-$. (I)

Here $x=s$, t, or u; and y is a second Mandelstam variable.

Complete threefold symmetry occurs only in scattering of like particles (π-π, N-N); unlike-particle scattering (π-N) has just twofold symmetry, conventionally between the s and u channels. Upon reduction from three- to twofold symmetry, $D\to A+S$; then theorem (I) is equivalent to remanence of the amplitude a^S at infinite energy. Closer inspection of this case shows that

for unlike particles the vanishing amplitudes correspond to odd isotopic spins in the t channel. (II)

This at once implies two well-known results:

(i) Pomeranchuk's theorem[1] on the equality of particle and antiparticle cross sections;

(ii) restriction of the number of independent amplitudes[2] to $N^S=[l]+1$.

Here $[x]$ is the largest integer contained in x, and l is the smaller of the two particle isotopic spins.

For like-particle scattering, one expects (I) and (II) still to be valid but supplemented by further restrictions; specifically, $N^D\leqslant N^S$ of the twofold amplitudes A and S will combine into threefold representation D, for which $|a^-/a^+|\to$ constant in the limit of infinite energy. But by (II), all the $a_t^-\to 0$ in the limit of infinite energy, so that all the corresponding a_t^+ do also, or hence the a^D do not contribute in the infinite limit. We thus infer a stronger result:

Only the totally symmetric amplitude a^S *contributes as* $x\to\infty$, $y=0^-$. (III)

For unlike particles (III) is equivalent to (I), but for like particles it contains the additional statement that a^D vanishes in the limit as well as a^A. This has the corollary:

(iii) Like-particle cross sections in the infinite limit are generally independent of isotopic spin only in case $l\leqslant\frac{1}{2}$.

Of course the immediately accessible case of N-N scattering fulfills this special condition.

The question arises of the relation between N^D and N^S. For integer l it appears that $N^D=N^S-1$; for half-integer l, $N^D=N^S$. In order to preserve any nonvanish-

[1] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 34, 725 (1958) [English transl.: Soviet Phys.—JETP 7, 499 (1958)].
[2] M. E. Rose and C. N. Yang, J. Math. Phys. 3, 106 (1962); D. Amati, J. Prentki, and A. Stanghellini, Nuovo Cimento 26, 1003 (1962).

ing cross section, we must associate to all half-integer l particles one independent internal index also of spinor character. This amounts to the "correlation[3] between real and charge space statistics." The combination of the two spinor indices then leads to one amplitude of type a^S. We thus conjecture the following:

Like-particle scattering has only one nonvanishing forward amplitude at infinite energies. (IV)

What one means by "like particles" depends on how inclusive is the symmetry classification used. Isotopic spin is the least inclusive, and SU(3) represents a considerable generalization. Extension[4,5] to SU(6) promises to incorporate various real spin varieties into supermultiplets, so that "like" particles could be all bosons or all fermions. If even this distinction disappears in some high enough symmetry, one has arrived at a universal limiting amplitude for all elastic two-body processes; note that it still satisfies corollary (iii) and cannot be so naively interpreted as the exchange of a "universal particle" with parameters of the vacuum.

Although theorems (I) to (IV) are in decreasing order of reliability, we shall use them all in the examples and applications of the following sections.

2. NOTATION AND DEFINITIONS

The standard two-body diagram is shown in Fig. 1, for which the associated variables are

$$\begin{aligned} s&=(p_1+p_2)^2=(p_3+p_4)^2,\\ t&=(p_1+p_3)^2=(p_2+p_4)^2,\\ u&=(p_1+p_4)^2=(p_2+p_3)^2. \end{aligned} \tag{2}$$

Indices (of isotopic spin) α, β, γ, δ are, respectively, associated with p_1, p_2, p_3, p_4. Consider the s channel:

$$a^{\alpha\beta\gamma\delta}(stu)=\sum_{I(s)}P_{I(s)}{}^{\alpha\beta\gamma\delta}a_s{}^I(stu), \tag{3}$$

where P is a projection operator and $\sum$ runs over all isotopic spin values possible in the s channel. The $a_s{}^I$ are simple scalar quantities; for both P and I, we must specify the defining channel for I, hence $I(s)$. For any channel

$$P_{I(s)}{}^{\epsilon\varphi\gamma\delta}P_{I'(s)}{}^{\gamma\delta\alpha\beta}=\delta_{II'}P_{I(s)}{}^{\epsilon\varphi\alpha\beta}, \tag{4a}$$

$$P_{I(s)}{}^{\alpha\beta\gamma\delta}P_{I(s)}{}^{\gamma\delta\alpha\beta}=P_{I(s)}{}^{\alpha\beta\alpha\beta}=(2I+1), \tag{4b}$$

$$\sum_{I(s)}P_{I(s)}{}^{\gamma\delta\alpha\beta}=\delta^{\gamma\alpha}\delta^{\delta\beta}. \tag{4c}$$

In more compact notation

$$P_{I(s)}\cdot P_{I'(s)}=(2I+1)\delta_{II'}, \tag{5a,b}$$

$$\sum P_{I(s)}=1_s, \tag{5c}$$

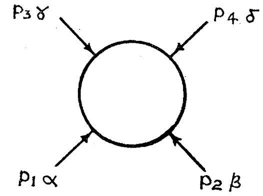

FIG. 1. Two-body scattering diagram.

where 1_s is an identity operator for isotopic spin in the s channel.

Now suppressing the individual indices α, β, γ, δ, we have

$$a(stu)=\sum_{I(s)}P_{I(s)}a_s{}^I=\sum_{J(t)}P_{J(t)}a_t{}^J=\sum_{K(u)}P_{K(u)}a_u{}^K. \tag{6}$$

To simplify notation further, we adopt throughout the convention in Eq. (6): I, J, K refer to isotopic spins in the respective s, t and u channels. By application of Eq. (5),

$$\begin{aligned} a^J&=\{(2J+1)^{-1}\sum_I P^J\cdot P^I\}a^I,\\ a^K&=\{(2K+1)^{-1}\sum_J P^K\cdot P^J\}a^J,\\ a^I&=\{(2I+1)^{-1}\sum_K P^I\cdot P^K\}a^K. \end{aligned} \tag{7}$$

The quantities in curly brackets are crossing matrices, computable by standard methods. From another point of view, however, they clearly provide representations of the permutation operators $\mathcal{P}_{ts}$, $\mathcal{P}_{ut}$, $\mathcal{P}_{su}$. These representations are generally reducible, for the matrices in Eq. (7) are all square and of dimension $(2l+1)$, where l is the minimum isotopic spin of any particle in the reaction. Accordingly, we seek to perform the reduction of the crossing matrices in Eq. (7) to $S+A+D$ in cases of physical interest and to exhibit the corresponding eigenvectors.

3. EXAMPLES OF EXCHANGE SYMMETRY

(a) Case of like particles, integer l. The special simplicity here is that I, J, and K all run over identical values from 0 to $2l$, so the situation is the same in every channel. For pion-pion scattering $l=1$, and if we write a column matrix

$$[a^I]=\begin{pmatrix}a^0\\a^1\\a^2\end{pmatrix},$$

it is clear that

$$\mathcal{P}_{ut}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}. \tag{8}$$

The well-known crossing matrix for this case is

$$\mathcal{P}_{ts}=\begin{pmatrix}\frac{1}{3}&1&5/3\\ \frac{1}{3}&\frac{1}{2}&-\frac{5}{6}\\ \frac{1}{3}&-\frac{1}{2}&\frac{1}{6}\end{pmatrix}, \tag{9}$$

[3] D. C. Peaslee, J. Math. Phys. 4, 910 (1963), and earlier references quoted there.
[4] A. Pais, Phys. Letters 13, 175 (1964).
[5] D. Neville, Phys. Rev. 132, 844 (1963).

and correspondingly

$$\mathcal{P}_{su}=\begin{bmatrix} \frac{1}{3} & -1 & 5/3 \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}. \tag{10}$$

Upon transforming to the basis

$$[B]=\begin{bmatrix} a^S \\ a_s^+ \\ a_s^- \end{bmatrix}=\begin{bmatrix} a^0+2a^2 \\ \frac{2}{3}a^0-(5/3)a^2 \\ \sqrt{3}a^1 \end{bmatrix}, \tag{11}$$

we obtain the desired block structure,

$$\mathcal{P}_{ts}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}, \quad \mathcal{P}_{ut}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathcal{P}_{su}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ 0 & -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}. \tag{12}$$

The function a^S is the completely symmetric eigenfunction on any exchange $s\leftrightarrow t\leftrightarrow u\leftrightarrow s$ and corresponds to elastic π^0-π^0 scattering. a_s^+ and a_s^- in Eq. (11) are the two components of the representation D, being, respectively, $+$ and $-$ under the s-preserving exchange $\mathcal{P}_{ut}$. Because of the symmetry, this situation is repeated in both other channels: the same $a^S=a^0+2a^2$ in each channel, and $a_t^D=\mathcal{P}_{ts}a_s^D$, etc.

If we generalize from simple isotopic spin to the full SU_3 symmetry, the scattering of bosons ⑧ on themselves has

$$\mathcal{P}_{ut}=\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}, \tag{13}$$

for the column vector

$$[a]=\begin{bmatrix} a^1 \\ a^8 \\ a^{8'} \\ a^{27} \\ a^{10} \end{bmatrix}, \tag{14}$$

where a^{10} is the amplitude for ⑩$+\overline{⑩}$, the only combination to occur in this symmetric situation. The crossing matrix[6] for Eq. (14) is

$$\mathcal{P}_{ts}=\begin{bmatrix} \frac{1}{8} & 1 & 1 & 27/8 & \frac{5}{2} \\ \frac{1}{8} & -\frac{3}{10} & \frac{1}{2} & 27/40 & -1 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{2} & -9/8 & 0 \\ \frac{1}{8} & \frac{1}{5} & -\frac{1}{3} & 7/40 & -\frac{1}{6} \\ \frac{1}{8} & -\frac{2}{5} & 0 & -9/40 & \frac{1}{2} \end{bmatrix}. \tag{15}$$

[6] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964).

Upon transformation to the basis

$$[B]=\begin{bmatrix} 5a^1+a^8+a^{27} \\ \hline a^1+\frac{1}{2}a^8-\frac{1}{3}a^{27} \\ (2/\sqrt{3})a^{8'} \\ \hline \frac{5}{2}a^1-a^8-\frac{1}{6}a^{27} \\ (2/\sqrt{3})a^{10} \end{bmatrix}, \tag{16}$$

Eq. (13) remains unchanged, but Eq. (15) reduces to block structure:

$$\mathcal{P}_{ts}=\left[\begin{array}{c|cc|cc} 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & -\frac{1}{2} & \frac{1}{2}\sqrt{3} & \cdot & \cdot \\ \cdot & \frac{1}{2}\sqrt{3} & \frac{1}{2} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \cdot & \cdot & \cdot & \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{array}\right], \tag{17}$$

the dots representing zeros. This is the reduction ⑧×⑧$=S+D+D'$.

(b) Case of unlike particles. Consider first $l=\frac{1}{2}$, and let $L>\frac{1}{2}$ be the other isotopic spin. Complete symmetry among the values of I, J, K no longer obtains; by convention we take I, $K=L\pm\frac{1}{2}$, $J=0$, 1. The crossing matrix is still square, and the basis vectors in the s and u channels correspond; but no *a priori* correspondence is imposed on the basis vector in the t channel. In the s or u channel the basis

$$[B]=(2L+1)^{-1}\begin{pmatrix} (L+1)a^{L+1/2}+La^{L-1/2} \\ \pm[L(L+1)]^{1/2}(a^{L+1/2}-a^{L-1/2}) \end{pmatrix}, \tag{18}$$

corresponds to

$$\mathcal{P}_{su}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

In the t channel the same representation for $\mathcal{P}_{su}$ corresponds to

$$[B]=\begin{pmatrix} a^0 \\ \pm\sqrt{3}a^1 \end{pmatrix}. \tag{20}$$

The numerical factors are arranged to make $a_{\alpha\beta\gamma\delta}a_{\gamma\delta\alpha\beta}^+$ constants in all channels. We can now equate the basis vectors in Eqs. (18) and (20), whence

$$\mathcal{P}_{ts}=\mathcal{P}_{ut}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}$$

This identifies the representations S and A:

$$\begin{aligned} a^S &= (2L+1)^{-1}\,[(L+1)\,a_s^{L+1/2}+La_s^{L-1/2}]=a_t^0 \\ &= (2L+1)^{-1}\,[(L+1)\,a_u^{L+1/2}+La_u^{L-1/2}], \\ a^A &= \pm(2L+1)^{-1}[L(L+1)]^{1/2}(a_s^{L+1/2}-a_s^{L-1/2})=\sqrt{3}a_t^1 \\ &= \pm(2L+1)^{-1}[L(L+1)]^{1/2}(a_u^{L+1/2}-a_u^{L-1/2}). \end{aligned} \tag{22}$$

The uncertainty of sign in a^A is of no concern: The identification of a^S is unequivocal and after a^S is removed from a 2×2 representation of the three-element permutation, the remainder must be a^A.

For the general case of $l<L$ the only symmetry imposed *ab initio* is again between s and u channels; that is, there occurs a reduction from threefold to twofold symmetry. Correspondingly, the representation D is absent, the a^+ and a^- being subsumed under a^S and a^A. Counting the $+$ and $-$ eigenvalues of $\mathcal{P}_{su}$ (most conveniently done in the s channel for half-integer l and in the t channel for integer l) yields the number of independent a^S:

$$N^S=[l]+1, \tag{23}$$

where $[x]$ is the largest integer contained in x. This has previously[2] been derived as a corollary to Pomeranchuk's theorem[1]; in Sec. 4 we show that the necessary conditions for Eq. (23) are slightly weaker than for Pomeranchuk's theorem. The reason is essentially that the Pomeranchuk theorem holds for both $l<L$ and $l=L$, while Eq. (23) is only for $l<L$. In case $l=L$, Eq. (23) is replaced by a much stronger restriction, which requires even more conditions than Pomeranchuk's theorem.

(c) Case of $l=L=\frac{1}{2}$. This is again a situation where a prescribed relation exists among basis vectors and threefold symmetry applies. Take

$$[B]=\begin{pmatrix}\sqrt{3}a^1\\ a^0\end{pmatrix}, \tag{24}$$

in all channels. The basis vector (24) in the s channel has

$$\mathcal{P}_{ut}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{25a}$$

and the crossing matrix is

$$\mathcal{P}_{su}=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}. \tag{25b}$$

In this case we must associate the amplitudes a^0 and a^1 with the single representation D. An immediate question is the absence of representation S; it occurs in all other cases and according to theorem (III) must always be present, because it represents the total cross section in the high-energy limit.

Representation S can be supplied by requiring that all particles of isotopic spin $\frac{1}{2}$ (by extension, half-odd integer) have one other half-odd spin associated with them. The direct product $D\times D'$ then contains S. The second spin $\frac{1}{2}$ can belong to another charge index as in the K-meson case; then the meson is really a vector rather than a spinor in charge space, as implied by the association of K with π in R_7, G_2, and SU_3 classification schemes. More interesting is the nucleon case where the second spin is in real space; the necessity for S symmetry in the amplitude then implies the correlation of real and charge space statistics.[3] This postulate and Theorem III derive some mutual support from their close connection.

It is convenient to write down the basis vectors for product $D\times D'$ in the case of two spins $\frac{1}{2}$,

$$\begin{aligned} a^S&=a^{++}+a^{--}=3a^{11}+a^{00},\\ a^{D+}&=a^{++}-a^{--}=3a^{11}-a^{00},\\ a^{D-}&=a^{+-}+a^{-+}=\sqrt{3}(a^{10}+a^{01}),\\ a^A&=a^{+-}-a^{-+}=\sqrt{3}(a^{10}-a^{01}).\end{aligned} \tag{26}$$

Of course a second spinor index must also be associated with the $l=\frac{1}{2}$ in Sec. (b), although it was not made explicit. Its influence is apparent in Eqs. (19) and (20), however: The exchange symmetries are the opposite of those for a pure isotopic spin $\frac{1}{2}$, implying the presence of other indices in the exchange.

4. THE PHASE REPRESENTATION

The phase representation of a function with two independent variables has been discussed by Sugawara and Nambu,[7] and we simply adopt their results. The Mandelstam amplitude is written

$$a(stu)=P_1(stu)/P_2(stu)Q(stu), \tag{27}$$

where P_1, P_2 are finite polynomials and Q is a phase function having no zeros or poles except at infinity. This form has the advantage of simple behavior as $s\to\infty$, $t\leqslant 0$ remaining fixed, so that $u\to-\infty$. Then, provided that

$$\int^{\infty}\left[\frac{\delta(s,t)-\delta(s=\infty,t)}{s}\right]ds=(\text{convergent}), \tag{28}$$

with a similar condition on $\delta(u,t)$,

$$Q(stu)\xrightarrow[\substack{s\to\infty,\\ u\to-\infty,\\ t=\text{const}}]{}\beta(t)[s_0/(s_0-s)]^{\delta(s=\infty,t)/\pi}\times[u_0/(u_0-u)]^{\delta(u=\infty,t)/\pi}. \tag{29}$$

Here $s_0>0$ is the threshold value at which the physical phase $\delta(s,t)$ first becomes nonvanishing in the s channel, and likewise for u_0. For fixed t, the function $\beta(t)$ is just a constant normalized to $\beta(0)=1$.

In accord with experimental indications for strong interactions, we take $\delta(x=\infty,y=0)=\pi/2$ for all forward scattering amplitudes in the limit of infinite energy any $\pm$ signs being absorbed into P_1/P_2. Then, except for inconsequential constant factors involving s_0 and u_0, Eq. (29) becomes

$$Q(s,t=0,u)\to is^{-1}\quad\text{as}\quad s\to\infty. \tag{30}$$

Equation (30) is a basic assumption of theorem I, which therefore does not encompass Coulomb scattering or very weak interactions like ν-e scattering.

[7] M. Sugawara and Y. Nambu, Phys. Rev. **131**, 2335 (1963); **132**, 2724 (1963).

By the original definition of the phase representation, only the imaginary part of $\ln Q$ is unique, variations in the real part of $\ln Q$ being absorbed in P_1/P_2. The symmetry of Q is therefore specified by that of $\mathrm{Im}\{\ln Q\}$: namely S for the forward scattering amplitude in the high-energy limit. This fixing of the permutation symmetry of Q is really arbitrary and represents a degree of freedom inherent in the phase representation; we have made the choice of greatest convenience, so that the permutation symmetry of a will be given by P_1 (we can take P_2 to be S without loss of generality). It should be emphasized that we have now used up all the arbitrariness available in the phase representation, so that the details of P_1 are completely significant.

A totally symmetric P_1 must have the form

$$P^S=\sum_{lmn} a_{lmn}(s+t+u)^l(st+tu+us)^m(stu)^n$$
$$=\sum_{mn} a_{mn}(st+tu+us)^m(stu)^n\,, \qquad (31)$$

since $(s+t+u)^l=\text{const.}$ in this case. The simplest antisymmetric polynomial is

$$P_0{}^A=(s-t)(s-u)(t-u)\,, \qquad (32)$$

and it is readily shown that all antisymmetric polynomials are of the form $P^A=P^SP_0{}^A$. From this fact and Eqs. (31) and (32) it is clear that as $s\to\infty$, P^S and P^A behave, respectively, as even and odd powers of s; in combination with Eq. (30) this means that P^S may contribute to the dominant forward scattering amplitude which goes to s, but P^A cannot.

The two-dimensional representation D has two basic polynomial pairs,

$$P_0{}^+=2s-(t+u)\,,\quad P_0{}^-=\sqrt{3}(t-u)\,, \qquad (33a)$$

$$P_e{}^+=2s^2-(t^2+u^2)\,,\quad P_e{}^-=\sqrt{3}(t^2-u^2)\,. \qquad (33b)$$

Any polynomial of symmetry D can be expressed as a symmetric polynomial times $P_0{}^D$ plus another symmetric polynomial times $P_e{}^D$. The term containing $P_0{}^D$ is eliminated in the same way as P^A, since it goes as an odd power when $s\to\infty$.

Elimination of P^A alone establishes theorem I for the case $l<L$, as well as the formula of Eq. (23); the arguments leading to Eq. (23) yield theorem II. Elimination of both P^A and $P_0{}^D$ is sufficient to establish Pomeranchuk's theorem[6] on the equality of total cross sections for particle and antiparticle on any target in the high-energy limit. In the s channel this is equivalent to symmetry under $s\leftrightarrow u$, with $t\to 0^-$; this symmetry is inherent for P^S and obtains under the condition $s\to -u\to\pm\infty$ for $P_e{}^D$ in Eq. (33b).

The question of $P_e{}^D$ remains only for the case $l=L$; but in a sufficiently high-energy limit differences of rest mass or even charge cannot serve to distinguish particles. There is then no way of separating forward from backward scattering in any channel, a distinction inherent in $P_e{}^D$; e.g., $P_e{}^-$ changes sign on $t\leftrightarrow u$. A more formal way to see that $P_e{}^D$ is physically inappropriate is to note an inescapable ambiguity of sign in the ratio $P_e{}^+/P_e{}^-\to\text{const.}$ in the high-energy unit. But if the ratio should be negative for forward scattering, one of the amplitudes will be antiunitary.

This leaves only P^S for the case $l=L$, and we arrive at theorem III. Theorem IV is merely extrapolated from the examples below.

5. APPLICATIONS

(a) Nucleon-Nucleon Scattering

Here the real and isotopic spins must be combined as in Eq. (26). The amplitudes a^{00} then satisfy theorem I as follows:

$$a^{00}\to\tfrac{1}{2}a^S\,,\quad a^{11}\to\tfrac{1}{6}a^S\,,\quad a^{01},\,a^{10}\to 0\,, \qquad (34)$$

for forward scattering in the high-energy limit. The corresponding cross sections all tend to equality,

$$\sigma_{pp}\sim\tfrac{3}{4}a^{11}+\tfrac{1}{4}a^{01}\to\tfrac{1}{8}a^S\,,$$
$$\sigma_{pn}\sim\tfrac{3}{8}a^{11}+\tfrac{3}{8}a^{10}+\tfrac{1}{8}a^{01}+\tfrac{1}{8}a^{00}\to\tfrac{1}{8}a^S\,, \qquad (35)$$

with identical limits for $\sigma_{\bar{n}p}$ and $\sigma_{\bar{p}p}$, respectively. These are two more of Pomeranchuk's theorems.[8,9] It would be of interest to see experimentally whether nucleon-nucleon cross sections σ^{01} and σ^{10} vanish at extreme energies.

(b) Pion-Nucleon Scattering

Let $a_t{}^0=a^S$; then for infinite energy in the s or u channel

$$a^{1/2}\to a^{3/2}\to a^S\,, \qquad (36)$$

which is the Okun'-Pomeranchuk theorem.[9] In the t channel we have not made the real spin index explicit for the $N\bar{N}$ side, although isotopic spin has been treated correctly. Since the real spin amplitudes a^1 and a^0 exchange roles on passing from NN to $N\bar{N}$, the nonvanishing amplitudes in the $N\bar{N}$ channel are a^{10} and a^{01} from Eq. (34); the statement of Eq. (36) is that only a^{10} contributes to $N\bar{N}\to\pi\pi$. This does not, however, imply any specific relationship between the a^S of Eqs. (34) and (36).

[8] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **30**, 423 (1956) [English transl.: Soviet Phys.—JETP **3**, 306 (1956)].

[9] L. B. Okun' and I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **30**, 424 (1956) [English transl.: Soviet Phys.—JETP **3**, 307 (1956)].

(c) Contradiction to Universal Particle Exchange

In[10] each of the examples above, the total cross section turns out independent of isotopic spin in the high-energy limit. Although this is not a general feature (corollary iii), it has given rise to the postulate of universal particle exchange (UPE): All reactions in the high-energy limit are assumed to be dominated by exchange of a universal particle with parameters of the vacuum, in particular $I=0$. The present approach contradicts this assumption, the simplest counterexample being π-π scattering,[11]

$$a^0 \to (5/9)a^S, \quad a^1 \to 0, \quad a^2 \to (2/9)a^S. \qquad (37)$$

Another present departure from UPE is the apparent lack of any special relation among the various a^S in Eqs. (35)–(37). Note that Eqs. (36) and (37) imply a vanishing contribution of ρ exchange to π-π and π-N scattering in the high-energy limit. By extension, this implies that γ-N scattering is pure isoscalar in the high-energy limit.[12]

(d) Application of π-π Formulas

The assignments in Eq. (37) are illuminated by an interesting calculation of Kawaguchi,[13] where the process $\pi+N \to N+2\pi$ is considered as arising entirely from π-π knock-on. His Eq. (2.14) gives the various cross sections in terms of four amplitudes:

$$\begin{aligned}
&\sigma(\pi^+ + p \to \pi^+ + \pi^+ + n) = \tfrac{4}{5}|a_{3/2}(s)|^2,\\
&\sigma(\pi^+ + p \to \pi^+ + \pi^0 + n) = \tfrac{1}{5}|a_{3/2}(s)|^2 + |a_{3/2}(a)|^2,\\
&\sigma(\pi^- + p \to \pi^- + \pi^+ + p) = \tfrac{1}{5}|a_{3/2}(s)|^2 + |\tfrac{1}{3}a_{3/2}(a) + \tfrac{2}{3}a_{1/2}(a)|^2,\\
&\sigma(\pi^- + p \to \pi^- + \pi^+ + n) = |\tfrac{1}{3}(\tfrac{2}{5})^{1/2}a_{3/2}(s) - \tfrac{2}{3}a_{1/2}(s)|^2 + |\tfrac{1}{3}\sqrt{2}a_{3/2}(a) - \tfrac{1}{3}\sqrt{2}a_{1/2}(a)|^2,\\
&\sigma(\pi^- + p \to \pi^0 + \pi^0 + n) = |\tfrac{2}{3}(\tfrac{1}{5})^{1/2}a_{3/2}(s) + \tfrac{1}{3}\sqrt{2}a_{1/2}(s)|^2.
\end{aligned} \qquad (38)$$

Here the subscript denotes the total isotopic spin of the system; the argument a denotes antisymmetric ($I=1$) states of the two final pions; and s denotes symmetric states, with $I=2$ for $a_{3/2}(s)$ and $I=0$ for $a_{1/2}(s)$. If the partial cross sections for the $\pi \to 2\pi$ process obey the same equality as given in Eq. (36) for the total cross sections, we expect

$$\begin{aligned}
\sigma(\pi^+ + p) &= |a_{3/2}(s)|^2 + |a_{3/2}(a)|^2 = \sigma(\pi^- + p)\\
&= \tfrac{1}{3}|a_{3/2}(s)|^2 + \tfrac{1}{3}|a_{3/2}(a)|^2\\
&\quad + \tfrac{2}{3}|a_{1/2}(s)|^2 + \tfrac{2}{3}|a_{1/2}(a)|^2,
\end{aligned} \qquad (39)$$

or hence [Eq. (2.15) of Ref. 10],

$$|a_{3/2}(s)|^2 + |a_{3/2}(a)|^2 = |a_{1/2}(s)|^2 + |a_{1/2}(a)|^2. \qquad (40)$$

A number of conditions for satisfying Eq. (40) are discussed in Ref. 13; the author favors $a(s)=0$ and $a_{3/2}(a) = a_{1/2}(a)$, on the grounds that all cross sections for nucleon charge exchange vanish. Equation (37) implies on the other hand that

$$a(a)=0, \quad |\tfrac{1}{5}\sqrt{2}a_{3/2}(s)|^2/|a_{1/2}(s)|^2 = a^2/a^0 = \tfrac{2}{5}, \qquad (41)$$

or hence $|a_{3/2}(s)| = |a_{1/2}(s)|$. The factor $\frac{1}{5}\sqrt{2}a_{3/2}(s)$ occurs because of the normalization adopted in Ref. 13. Conditions (41) also satisfy Eq. (40), but do *not* imply the vanishing of nucleon charge exchange: This is another feature of the simplest situation (π-N scattering) that does not extrapolate to more complicated ones under the present approach.

(e) K-K Scattering

Simple isotopic spin conservation and Pomeranchuk's theorem on antiparticles are sufficient to ensure that the forward scattering amplitudes at infinite energy satisfy

$$\begin{aligned}
a(K^+K^+) = a(K^+K^-) = a(K^-K^-) &= a(K^0K^0)\\
&= a(K^0\bar{K}^0) = a(\bar{K}^0\bar{K}^0), \qquad (42a)
\end{aligned}$$

$$a(K^+K^0) = a(K^+\bar{K}^0) = a(K^-\bar{K}^0) = a(K^-K^0). \qquad (42b)$$

There are at most two amplitudes which can be related to the a^{11} and a^{00} of Eq. (26). Theorem III then implies a relation between these amplitudes, readily found to be

$$a(KK) \to a^S, \qquad (43)$$

a constant for all elastic K-K forward amplitudes, independent of isotopic spin or hypercharge as expected whenever $l=\frac{1}{2}$.

(f) Boson Octet-Octet Scattering

In the limit of infinite energy where rest-mass differences are negligible, we may assume boson-boson scattering to follow SU_3 octet symmetry. Vanishing of all but the first line of Eq. (16) implies

$$\begin{aligned}
&a^1 : a^8 : a^{27} = 5:8:27,\\
&a^{10} + a^{\overline{10}} = a^{8'} = 0.
\end{aligned} \qquad (44)$$

We may then use a table[14] of Clebsch-Gordan coeffi-

[10] If UPE is in fact an incorrect idea, it seems hardly fair to name it after the author of Ref. 8, who did not suggest it.

[11] Note that a^S always contains some $I=0$ component, so that no contradiction is implied to the theorems of L. L. Foldy and R. F. Peierls, Phys. Rev. **130**, 1585 (1963), or of D. Amati, L. L. Foldy, A. Stanghellini, and L. Van Hove, Nuovo Cimento **32**, 1685 (1964).

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cients for ⑧×⑧ to obtain ratios of various physical scattering amplitudes:

$$a(KK)=a^{3/2}(K\pi)=a^{1/2}(K\pi)=a^2(\pi\pi)$$
$$=a(K\eta)=a(\pi\eta)=a^S, \quad (45)$$
$$a^1(\pi\pi)=0, \quad a^0(\pi\pi)=\tfrac{5}{2}a^S, \quad a(\eta\eta)=\tfrac{3}{2}a^S.$$

Equation (45) relates the a^S of Eqs. (37) and (43) with those for several other processes; in the present approach such relations are not universal but result from the specific assumption of a higher symmetry.

(g) Boson-Fermion Octet Scattering

Although certain objections may be raised[3] against the use of octet symmetry for fermion-boson interactions, we can apply the formulas without difficulty; in this case, only twofold symmetry obtains, and there are three possible independent cross sections in the s channel:

$$\begin{aligned} &a(\eta N)\\ &a^0(\bar{K}N)=\tfrac{1}{2}[3a^1(KN)-a^0(KN)],\\ &a^1(\bar{K}N)=\tfrac{1}{2}[a^1(KN)+a^0(KN)]\\ &\qquad=a^{3/2}(\pi N)=a^{1/2}(\pi N).\end{aligned} \quad (46)$$

The separate Eqs. (46) are sufficient to satisfy Pomeranchuk's theorem *without* necessarily implying isotopic spin independence for KN or $\bar{K}N$ scattering. In this case isotopic spin is only part of a more inclusive symmetry. The appearance in high-energy data of a familial relation between πN and KN scattering has been pointed out.[15]

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Boson-Pole Model in K-Meson and η-Meson Decays*

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The direct term of the decay mode $K\rightarrow\pi+\pi+\gamma$ and the decay amplitudes $K\rightarrow\pi+\pi+\pi$ and $\eta\rightarrow\pi+\pi+\pi$ are obtained on the basis of the boson-pole model to all orders in the strong interaction, provided the interaction (except for the weak vertex) is SU_3-invariant. The weak vertex is assumed to transform as K°. Then all $K\rightarrow\pi+\pi+\gamma$ modes, except $K_2^0\rightarrow\pi^+\pi^-\gamma$, and all $K\rightarrow\pi+\pi+\pi$ and $\eta\rightarrow\pi+\pi+\pi$ modes, except $(\eta|\pi^+\pi^-\pi^0)$, are shown to vanish. It is concluded that the boson-pole model together with unitary symmetry is untenable for $K\rightarrow 3\pi$ decay modes.

1. INTRODUCTION

THE boson-pole model[1] has been used to compute the direct term of the $K\rightarrow 2\pi+\gamma$ mode,[2] $K\rightarrow 3\pi$ and $\eta\rightarrow 3\pi$ modes,[3,4] as well as other processes. In this model, the initial boson P_1 converts into another boson P_2 by a weak vertex $P_1\rightarrow P_2$, which then turns into the final states $P_2\rightarrow P_3+P_4+\gamma$ (radiative mode) or $P_2\rightarrow P_3+P_4+P_5$ (3π mode) by electromagnetic and strong interactions, i.e., for example,

$$P_1\rightarrow P_2\rightarrow P_3+P_4+\gamma.$$

There is another class of diagrams in which the weak vertex follows the electromagnetic and strong interactions, i.e.,

$$P_1\rightarrow P_2+P_3+\gamma\rightarrow P_4+P_3+\gamma.$$

The part of the diagram that depends on the electromagnetic and strong interactions can be regarded in unitary space as $\gamma+P\rightarrow P+P$ and $P+P\rightarrow P+P$, i.e., two octets transforming into two other octets.[5,6] The photon transforms as a singlet ϕ^0 plus the third component of a vector ρ^0, i.e.,

$$\gamma=-\tfrac{1}{2}\phi^0-\tfrac{1}{2}\sqrt{3}\rho^0.$$

It will be shown later that because of conservation of momentum, it is a reasonable approximation to combine the two types of diagrams when the bosons have their respective physical masses.

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